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DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY, ZARIA
2023/2024 FIRST SEMESTER EXAMINATIONS- MATH 317

Attempt ANY FOUR Questions **Time Allowed 2hrs.**

Q1. (a) Determine the first five Picard iterates for the IVP $y' = 2e^x - y$ subject to the initial condition $y(0) = 1$.

(b) Use a general 3×3 matrix and show that the Jacobi matrix iteration scheme for a system of linear equation $Ax = B$ is given by $x^n = D^{-1}B - D^{-1}(L + U)x^{n-1}$.

Where $D, L,$ and U denote the Diagonal, Lower and Upper triangular matrices respectively.

Q2. (a) Using Runge-Kutta (4th Order) method, find. correct to 4 decimal places y_i, $i = 1,2,\dots,5$ if $\frac{dy}{dx} = 3x + y^2$ subject to the initial conditions $y(1) = 2$ and step size: $h = 0.2$.

(b) By performing only five iterations, solve the system of linear equations using the Successive Over Relaxation (SOR) method with an initial guess $[0, 0, 0]^T$ and relaxation factor $\beta = 1.1$.

Q3. Employing the Milne's Predictor-Corrector method, evaluate $y(x)$ for the IVP $\frac{dy}{dx} = x - y^2$ at $x = 0.8$ (correct to 4 decimal places) given that $y(0) = 0, y(0.2) = 0.02$, $y(0.4) = 0.0795$ and $y(0.6) = 0.1762$.

(b) Define Diagonal Dominant and determine whether or not the system of linear equations: $6x - y - z = 19$, $x + 2y + 6z = 22$ and $3x + 4y + z = 26$ is Diagonal Dominant. Hence obtain the approximate solution for the system using the Jacobi matrix iteration method.

Q4 (a) Determine the numerical solution, using the Tylor's series method, for $\frac{dy}{dx} = 1 + xy$ with an initial condition $x_0 = 0$ and $y_0 = 1$ (up to the 3rd power of x). Hence use your result to determine the value of $y(0.1)$.

(b) Obtain, using the Gauss-Seidel method, the approximate solution for the system of the equations

$x + 9y - 2z = 36$, $2x - y + 8z = 121$ and $6x + y + z = 107$.

Q5. (a) Find the exact solution for $\frac{dy}{dx} = xy$ subject to $x_0 = 0$ and $y_0 = 1$ and compare the results with the Picard's iterates at $x = 0.3$. What is the percentage error?

D. Raji

(b) Copy and complete the table below for the IVB $y' = 3x + \frac{y}{2}$ subject to the initial conditions $y(0) = 1$

x	Euler's method	Improved Euler's	Exact solution $y(x) = 13e^{\frac{x}{2}} - 6x - 12$
0.0			
0.1			
0.2			
0.3			
0.4			
0.5			

Q6 (a) Use Picard's method to obtain $y\ (0.2)$, $y\ (0.4)$ and $y\ (0.6)$ for the IVP $\frac{dy}{dx} = x + y$, with $y\ (0) = 1$.

(b) Using the iterates in (a) calculate $y\ (0.8)$ using Adam-Bashforth Predictor-Corrector method (correct to 4 decimal places).

MATH317: NUMERICAL ANALYSIS II, 2023/2024 SESSION.

(1a) Given the IVP. $y' = 2e^x - y$ subject to $y(0)=1$

Using Picard iteration method i.e $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$.

First iteration $n=1$, $y_0=1$, $x_0=0$

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx = 1+\int_0^x (2e^x - y_0)\,dx = 1+\int_0^x (2e^x - 1)\,dx = 1+\left[2e^x - x\right]_0^x$$

$$= 1+\left[2e^x - x - 2\right] = 1+2e^x - x - 2 = 2e^x - x - 1.$$

Second iteration, $n=2$, $y_1 = 2e^x - x - 1$, $y_0 = 1$

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx = y_0 + \int_0^x (2e^x - y_1)\,dx = 1+\int_0^x (2e^x - 2e^x + x + 1)\,dx$$

$$= 1+\int_0^x (x+1)\,dx = 1+\left(\frac{x^2}{2}+x\right|_0^x = 1+\left[\frac{x^2}{2}+x-0-0\right] = 1+\frac{x^2}{2}+x.$$

Third iteration $n=3$, $y_2 = 1+\frac{x^2}{2}+x$, $y_0=1$

$$y_3 = y_0 + \int_0^x f(x, y_2)\,dx = 1+\int_0^x (2e^x - y_2)\,dx = 1+\int_0^x \left(2e^x - 1 - \frac{x^2}{2} - x\right)dx$$

$$= 1+\left(2e^x - x - \frac{x^3}{6} - \frac{x^2}{2}\right|_0^x = 1+2e^x - x - \frac{x^3}{6} - \frac{x^2}{2} - 2 = 2e^x - x - \frac{x^3}{6} - \frac{x^2}{2} - 1$$

Fourth iteration $n=4$, $y_3 = 2e^x - x - \frac{x^3}{6} - \frac{x^2}{2} - 1$, $y_0 = 1$

$$y_4 = y_0 + \int_0^x f(x, y_3)\,dx = 1+\int_0^x (2e^x - y_3)\,dx = 1+\int_0^x \left(2e^x - 2e^x + x + \frac{x^3}{6} + \frac{x^2}{2} + 1\right)dx$$

$$= 1+\int_0^x \left(x + \frac{x^3}{6} + \frac{x^2}{2} + 1\right)dx = 1+\left(\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^3}{6} + x\right|_0^x = 1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^3}{6}+x.$$

Fifth iteration $n=5$, $y_4 = 1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^3}{6}+x$

$$y_5 = y_0 + \int_0^x f(x, y_4)\,dx = 1+\int_0^x \left(2e^x - 1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^3}{6} - x\right)dx = 1+\left[2e^x - x - \frac{x^3}{6} - \frac{x^5}{120} - \frac{x^4}{24} - \frac{x^2}{2}\right|_0^x$$

$$= 1+2e^x - x - \frac{x^3}{6} - \frac{x^5}{120} - \frac{x^4}{24} - \frac{x^2}{2} - 2 = 2e^x - \frac{x^5}{120} - \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{2} - x - 1$$

(1b) Given a system of linear equation $Ax = B$ (matrix) and $A = L+U+D$

Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad L = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \quad U = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Then, $Ax = B \Rightarrow (L+U+D)x = B \Rightarrow (L+U)x + Dx = B$

$$\Rightarrow Dx = B - (L+U)x \Rightarrow x = \frac{B-(L+U)x}{D} = \frac{B}{D} - \frac{(L+U)x}{D}$$

$$\Rightarrow x = BD^{-1} - D^{-1}(L+U)x = D^{-1}B - D^{-1}(L+U)x.$$

Hence, the iteration formular can be obtained as follows:

$$x^n = BD^{-1} - D^{-1}(L+U)x^{n-1}$$

(2a) $\frac{dy}{dx} = 3x + y^2$, $y(1) = 2$, $h = 0.2$ $y_i = 1, 2 \cdots 5$ iterations

Using Runge-Kutta (4th order)

$K_1 = hf(x_0, y_0)$, $K_2 = hf(x_0 + h/2, y_0 + k_1/2)$, $K_3 = hf(x_0 + h/2, y_0 + k_2/2)$, $K_4 = h(x_0 + h, y_0 + k_3)$

where $K = K_1 + 2(K_2 + K_3) + K_4$.

$\Rightarrow y(x) = y_0 + \frac{K}{6}$.

First iteration $i = 1$. $x_0 = 1$, $y_0 = 2$

$K_1 = 0.2f(3x_0 + y_0^2) = 0.2(3(1) + 2] = 0.2[3+2] = 0.2(5) = 1.0$

$K_2 = 0.2f(3(x_0 + h/2) + (y_0 + k_1/2)^2] \Rightarrow x_0 + h/2 = 1 + \frac{0.2}{2} = 1.1$ & $y_0 + \frac{k_1}{2} = 2 + \frac{1}{2} = 2.5$

$K_2 = 0.2(3(1.1) + (2.5)^2] = 0.2[3.3 + 6.25] = 0.2(9.55) = 1.91$

$K_3 = 0.2f(3(x_0 + h/2) + (y_0 + \frac{k_2}{2})^2] \Rightarrow x_0 + h/2 = 1.1$ and $y_0 + \frac{k_2}{2} = 2.955$

$K_3 = 0.2[3(1.1) + (2.955)^2] = 0.2[3.3 + 8.7320] = 2.41$

$K_4 = 0.2f(3(x_0 + h) + (y_0 + k_3)^2] = 0.2[3(1.2) + 4.41^2] = 0.2[23.0481) = 4.61$

$K = K_1 + 2(K_2 + K_3) + K_4 = 1.0 + 2(1.91 + 2.41) + 4.61 = 14.25$

$\Rightarrow y(1) = y_0 + \frac{K}{6} = 2 + \frac{14.25}{6} = 2 + 2.375 = \underline{\underline{4.375}}$

Second iteration New $y_0 = 4.375$, $x_0 = x_0 + h = 1 + 0.2 = 1.2$, $h = 0.2$.

$K_1 = 0.2f(3x_0 + y_0^2] = 0.2[3(1.2) + 4.375^2] = 4.548$

$K_2 = 0.2f(3(x_0 + \frac{k_1}{2}) + (y_0 + \frac{k_1}{2})^2]$ where $x_0 + h/2 = 1.3$ and $y_0 + \frac{k_1}{2} = 6.649$

$K_2 = 0.2(3(1.3) + 6.649^2] = 0.2(48.109) = 9.622$

$K_3 = 0.2f(3(x_0 + h/2) + (y_0 + \frac{k_2}{2})^2]$ where $x_0 + h/2 = 1.3$ and $y_0 + \frac{k_2}{2} = 9.186$

$K_3 = 0.2(3(1.3) + 9.186^2] = 0.2(88.2826) = 17.656$

$K_4 = 0.2f(3(x_0 + h) + (y + k_3)^2]$ where $x_0 + h = 1.4$ and $y + k_3 = 22.031$

$K_4 = 0.2(3(1.4) + 22.031^2] = 0.2(489.565) = 97.913$

$K = k_1 + 2(K_2 + K_3) + K_4 = 4.548 + 2(9.622 + 17.656) + 97.913 = 157.017$

$\Rightarrow y(2) = y_0 + \frac{K}{6} = 4.375 + \frac{157.017}{6} = 30.545$

Third iteration New $y_0 = 30.545$, $x_0 = x_0 + h = 1.2 + 0.2 = 1.4$, $h = 0.2$

$K_1 = 0.2f(3x_0 + y_0^2) = 0.2(3(1.4) + 30.545^2) = 0.2(937.197) = 187.439$

$K_2 = 0.2f(3(x_0 + h/2) + (y_0 + k_1/2)^2]$ where $x_0 + h/2 = 1.5$ and $y_0 + k_1/2 = 124.2645$

$K_2 = 0.2(3(1.5) + 124.2645^2) = 0.2(15446.16596) = 3089.2332$

$K_3 = 0.2f(3(x_0 + h/2) + (y_0 + k_2/2)^2]$ where $x_0 + h/2 = 1.5$ and $y_0 + k_2/2 = 1575.1616$

$K_3 = 0.2(3(1.5) + 1575.1616^2) = 0.2(2481138.566) = 496227.71$

$K_4 = 0.2f\left(3(x_0+h)+(y_0+k_3)^2\right)$ = where $x_0+h = 1.6$ and $y_0+k_3 = 496258.255$

$K_4 = 0.2(3(1.6)+496258.255^2) = 0.2(2.4627\times10^{11}) = 4.925\times10^{10}$.

$K = K_1 + 2(K_2+K_3)+K_4 = 187.439+2(3089.2332+496227.71)+4.925\times10^{10} = 4.9251\times10^{10}$

$y(3) = y_0 + \frac{k}{6} = 30.545 + \frac{4.9251\times10^{10}}{6} = 8208500030$

Forth iteration Now $y_0 = 8208500030$, $x_0 = x_0+h = 1.4+0.2 = 1.6$, $h=0.2$

$K_1 = 0.2f(3x_0+y_0^2) = 0.2(3(1.6)+8208500030^2) = 1.3476\times10^{17}$

$K_2 = 0.2f[3(x_0+h/2)+(y_0+K_1/2)^2]$ where $x_0+h/2 = 1.7$ and $y_0+K_1/2 = 6.738\times10^{16}$

$K_2 = 0.2(3(1.7)+(6.738\times10^{16})^2] = 0.2(4.540066\times10^{33}) = 9.080\times10^{32}$

$K_3 = 0.2f(3(x_0+h/2)+(y_0+K_2/2)^2]$ where $x_0+h/2 = 1.7$ and $y_0+K_2/2 = 4.54\times10^{32}$

$K_3 = 0.2(3(1.7)+(4.54\times10^{32})^2] = 0.2(2.0612\times10^{65}) = 4.122\times10^{64}$

$K_4 = 0.2f(3(x_0+h)+(y_0+K_3)^2]$ where $x_0+h = 1.8$ and $y_0+K_3 = 4.1224\times10^{64}$

$K_4 = 0.2(3(1.8)+(4.1224\times10^{64})^2] = 0.2(3(1.8)+\text{Math error}) = \text{Math - error}$.

Hence, Can't be Continued.

(2b) For successive, we have,

$x_n = \frac{w}{a_1}(K_1 - a_2y_{n-1} - a_3Z_{n-1}) + (1-w)x^{n-1}$

$y_n = \frac{w}{b_2}(K_2 - b_1x_n - b_3Z_{n-1}) + (1-w)y^{n-1}$

$Z_n = \frac{w}{C_3}(k_3 - C_1x_n - C_2y_n) + (1-w)z^{n-1}$

$w = 1.1 > 1$ —— SOR initial guess $[0,0,0]^T$

There's no system of linear equation.

(3a) $\frac{dy}{dx} = x - y^2$ at $x = 0.8$ Given $y(0)=0$, $y(0.2)=0.02$, $y(0.4)=0.0795$ and $y(0.6)=0.1762$ $h=0.2$.

Using Milne's predictor - corrector method.

$y_4^P = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$ and $y_4^c = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$y_0' = x - y^2 = 0 - 0^2 = 0$, $y_1' = 0.2 - (0.02)^2 = 0.1996$

$y_2' = 0.4 - 0.0795^2 = 0.3937$, $y_3' = 0.6 - 0.1762^2 = 0.56895$

$y_4^P = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.56895)] = \frac{0.8}{3}(1.1434) = 0.3049$

and $y_4^c = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.56895) + 0.7070] = 0.0795 + \frac{0.2}{3}(3.3765)$
$= 0.3046$

where $y'_4 = 0.8 - 0.3049^2 = 0.7070$.

n	x_n	y_n	$y'_n = x - y^2$
0	0	0	0
1	0.2	0.02	0.1996
2	0.4	0.0795	0.3937
3	0.6	0.1762	0.56895
4	0.8	Predictor 0.3049	0.7040
		Corrector 0.3046	0.7072

(3b) Diagonal dominant matrix is a square matrix in which the size of each diagonal entry is greater than or equal to the sum of the sizes of the other (non-diagonal) entries in the same row or column.

If we have $m \times m$ matrix such that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

if $|a_{11}| \geq |a_{12}| + |a_{13}|$

$|b_{22}| \geq |b_{21}| + |b_{23}|$

$|c_{33}| \geq |c_{31}| + |c_{32}|$.

Now, to check whether the system of linear equation: $6x - y - z = 19$, $x + 2y + 6z = 22$ and $3x + 4y + z = 26$ is diagonal dominant. We do;

$$\left.\begin{aligned} 6x - y - z &= 19 \\ 3x + 4y + z &= 26 \\ x + 2y + 6z &= 22 \end{aligned}\right\} \text{Arrange}$$

$$\Rightarrow \begin{pmatrix} 6 & -1 & -1 \\ 3 & 4 & 1 \\ 1 & 2 & 6 \end{pmatrix}$$

$|6| \geq |-1| + |-1| \Rightarrow 6 \geq 2$

$|4| \geq |3| + |1| \Rightarrow 4 \geq 4$

$|6| \geq |1| + |2| \Rightarrow 6 \geq 3$

Hence, the system of the linear equation above is diagonally dominant $\square$.

Using the jacobi matrix iteration method:

$$Ax = b$$

$$\begin{pmatrix} 6 & -1 & -1 \\ 3 & 4 & 1 \\ 1 & 2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ 26 \\ 22 \end{pmatrix} \Rightarrow L = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and $D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ But $(L+U)D^{-1} = \begin{pmatrix} 0 & -1/4 & -1/6 \\ 1/2 & 0 & 1/6 \\ 1/6 & 1/2 & 0 \end{pmatrix}$

$$\Rightarrow L+U=\begin{pmatrix}0&-1&-1\\3&0&1\\1&2&0\end{pmatrix} \text{ and } D^{-1}=\begin{pmatrix}1/6&0&0\\0&1/4&0\\0&0&1/6\end{pmatrix} \; bD^{-1}=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}$$

Hence,

$$x^n = bD^{-1}-(L+U)D^{-1}x^{n-1} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}^n = bD^{-1}-(L+U)D^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix}^{n-1}$$

First iteration $n=1$ $x_0=0$, $y_0=0$, $z_0=0$

$$x^1 = bD^{-1}-(L+U)D^{-1}x^{1-1} \Rightarrow x^1 = bD^{-1}-(L+U)D^{-1}x^0$$

$$x^1=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}-\begin{pmatrix}0&-0.25&-0.1667\\0.5&0&0.1667\\0.1667&0.5&0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow x^1=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}$$

Second iteration $n=2$ $x_1=3.1667$, $y_1=6.5$ and $z_1=3.667$

$$x^2=bD^{-1}-(L+U)D^{-1}x^{2-1} \Rightarrow x^2=bD^{-1}-(L+U)D^{-1}x^1$$

$$\Rightarrow x^2=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}-\begin{pmatrix}0&-0.25&-0.1667\\0.5&0&0.1667\\0.1667&0.5&0\end{pmatrix}\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}$$

$$=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}-\begin{pmatrix}-2.2362\\2.1946\\3.7779\end{pmatrix}=\begin{pmatrix}5.4029\\4.3054\\-0.1112\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}\equiv\begin{pmatrix}x^2\\y^2\\z^2\end{pmatrix}=\begin{pmatrix}5.4029\\4.3054\\-0.1112\end{pmatrix}$$

Third iteration $n=3$, $x_2=5.4029$, $y_2=4.3054$ and $z_2=-0.1112$.

$$x^3=bD^{-1}-(L+U)D^{-1}x^{3-1} \Rightarrow x^3=bD^{-1}-(L+U)D^{-1}x^2$$

$$\Rightarrow x^3=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}-\begin{pmatrix}0&-0.25&-0.1667\\0.5&0&0.1667\\0.1667&0.5&0\end{pmatrix}\begin{pmatrix}5.4029\\4.3054\\-0.1112\end{pmatrix}$$

$$=\begin{pmatrix}3.1667\\6.5000\\3.6667\end{pmatrix}-\begin{pmatrix}-1.0579\\2.6830\\3.0534\end{pmatrix}=\begin{pmatrix}4.2246\\3.8170\\0.6133\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x_3\\y_3\\z_3\end{pmatrix}\equiv\begin{pmatrix}x^3\\y^3\\z^3\end{pmatrix}=\begin{pmatrix}4.2246\\3.8170\\0.6133\end{pmatrix}$$

forth iteration, $n=4$, $x_3=4.2246$, $y_3=3.8170$, $z_3=0.6133$

$$x^4 = bD^{-1} - (L+U)D^{-1}x^{4-1} \Rightarrow x^4 = bD^{-1} - (L+U)D^{-1}x^3$$

$$x^4 = \begin{pmatrix} 3.1667 \\ 6.5000 \\ 3.6667 \end{pmatrix} - \begin{pmatrix} 0 & -0.25 & -0.1667 \\ 0.5 & 0 & 0.1667 \\ 0.1667 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} 4.2246 \\ 3.8170 \\ 0.6133 \end{pmatrix}$$

$$= \begin{pmatrix} 3.1667 \\ 6.5000 \\ 3.6667 \end{pmatrix} - \begin{pmatrix} -1.0565 \\ 2.2145 \\ 2.6127 \end{pmatrix} = \begin{pmatrix} 4.2232 \\ 4.2855 \\ 1.0543 \end{pmatrix}$$

$$\Rightarrow x^4 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \equiv \begin{pmatrix} x^4 \\ y^4 \\ z^4 \end{pmatrix} = \begin{pmatrix} 4.2232 \\ 4.2855 \\ 1.0543 \end{pmatrix}$$

fifth iteration, $n=5$, $x_4=4.2232$, $y_4=4.2855$, $z_4=1.0543$

$$x^5 = bD^{-1} - (L+U)D^{-1}x^{5-1} \Rightarrow x^5 = bD^{-1} - (L+U)D^{-1}x^4$$

$$x^5 = \begin{pmatrix} 3.1667 \\ 6.5000 \\ 3.6667 \end{pmatrix} - \begin{pmatrix} 0 & -0.25 & -0.1667 \\ 0.5 & 0 & 0.1667 \\ 0.1667 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} 4.2232 \\ 4.2855 \\ 1.0543 \end{pmatrix}$$

$$= \begin{pmatrix} 3.1667 \\ 6.5000 \\ 3.6667 \end{pmatrix} - \begin{pmatrix} -1.2316 \\ 2.2874 \\ 2.8468 \end{pmatrix} = \begin{pmatrix} 4.3983 \\ 4.2126 \\ 0.8199 \end{pmatrix}$$

$$\Rightarrow x^5 = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \equiv \begin{pmatrix} x^5 \\ y^5 \\ z^5 \end{pmatrix} = \begin{pmatrix} 4.3983 \\ 4.2126 \\ 0.8199 \end{pmatrix}$$

Hence, the approximate solution of the system includes;
$x=4.3983$, $y=4.2126$ and $z=0.8199$ respectively.

4a) $\frac{dy}{dx} = 1 + xy$ with an initial condition $x_0=0$, $y_0=1$
Given the Taylor's Series method.

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots + \frac{(x-x_0)^{n-1}}{(n-1)!}y^{n-1}(x_0) + \cdots$$

Here; $x_0=0$, $y_0=1$ $\Rightarrow y'(x_0) = y'(0) = 1 + (0\times1) = 1$

$y''(x) = 0 + y + xy' \Rightarrow y''(x_0) = y''(0) = 1 + (0\times1) = 1$

$y'''(x) = y' + y' + xy'' = 2y' + xy'' \Rightarrow y'''(0) = 2(1) + (0\times1) = 2$

$y^{iv}(x) = 2y'' + y'' + xy''' = 3y'' + xy''' \Rightarrow y^{iv}(0) = 3(2) + (0\times2) = 6$

$y^{v}(x) = 3y''' + y''' + xy^{iv} = 4y''' + xy^{iv} \Rightarrow 4(2) + (0\times6) = 8$.

Now, Using the Taylor's method, we have; (Up to the third power i.e x^3)

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots + \frac{(x-x_0)^{n-1}}{(n-1)!}y^{n-1}(x_0) + \cdots$$

$$= 1 + (x-0)(1) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(2)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \cdots = 1 + x + \frac{x^2}{4} + \frac{2x^3}{6} = 1 + x + \frac{x^2}{4} + \frac{x^3}{3}$$

Hence, to determine the value of $y(0.1)$ we have;

$$y(0.1) = 1 + x + \frac{x^2}{4} + \frac{x^3}{3} = 1 + 0.1 + \frac{(0.1)^2}{4} + \frac{(0.1)^3}{3}$$

$$= 1 + 0.1 + \frac{0.01}{4} + \frac{0.001}{3} = 1.1 + 0.0025 + 0.00033 = 1.1028$$

Thus, the value of $y(0.1) = 1.1028$

4b) The system of equation; $x + 9y - 2z = 36$, $2x - y + 8z = 121$ and $6x + y + z = 107$

Using the Gauss-Seidel method, we first check whether the diagonal dominant is intact. Since it is not intact we rearrange.

$\Rightarrow 6x + y + z = 107$, $x + 9y - 2z = 36$ and $2x - y + 8z = 121$.

From eqn ①, ②, ③ we have;

$$x = \frac{1}{6}(107 - y - z), \quad y = \frac{1}{9}(36 - x + 2z) \ \& \ z = \frac{1}{8}(121 - 2x + y)$$

$$\Rightarrow x^n = \frac{1}{6}(107 - y^{n-1} - z^{n-1}), \quad y^n = \frac{1}{9}(36 - x^n + 2z^{n-1}) \ \& \ z^n = \frac{1}{8}(121 - 2x^n + y^n)$$

First iteration $n=1$ $x_0 = 0$, $y_0 = 0$, $z_0 = 0$

$$x^1 = \frac{1}{6}(107 - y^0 - z^0) = \frac{1}{6}(107 - 0 - 0) = \frac{107}{6} = 17.8333$$

$$y^1 = \frac{1}{9}(36 - 17.8333 + 2(0)] = \frac{1}{9}(36 - 17.8333) = 2.0185$$

$$z^1 = \frac{1}{8}(121 - 2(17.8333) + 2.0185) = \frac{1}{8}(87.3519) = 10.9190$$

$x_1 = 17.8333$, $y_1 = 2.0185$, $z_1 = 10.919$

Second iteration $n=2$, $x_1 = 17.8333$, $y_1 = 2.0185$, $z_1 = 10.919$

$$x^2 = \frac{1}{6}(107 - 2.0185 - 10.919) = \frac{1}{6}(94.0625) = 15.6771$$

$$y^2 = \frac{1}{9}(36 - 15.6771 + 2(10.919)] = \frac{1}{9}(42.1609) = 4.6845$$

$$z^2 = \frac{1}{8}(121 - 2(15.6771) + 4.6845) = \frac{1}{8}(94.3303) = 11.7913$$

Third iteration $n=3$, $x_2 = 15.6771$, $y_2 = 4.6845$, $z_2 = 11.7913$

$x^3 = \frac{1}{6}(107 - 4.6845 - 11.7913) = \frac{1}{6}(90.5242) = 15.0874$
$y^3 = \frac{1}{9}(36 - 15.0874 + 2(11.7913)) = \frac{1}{9}(44.4952) = 4.9439$
$z^3 = \frac{1}{8}(121 - 2(15.0874) + 4.9439)) = \frac{1}{8}(95.7691) = 11.9711$
forth iteration $n=4$, $x_3 = 15.0874$, $y_3 = 4.9439$, $z_3 = 11.9711$
$x^4 = \frac{1}{6}(107 - 4.9439 - 11.9711) = \frac{1}{6}(90.085) = 15.0142$
$y^4 = \frac{1}{9}(36 - 15.0142 + 2(11.9711)]] = \frac{1}{9}(44.928) = 4.9920$
$z^4 = \frac{1}{8}(121 - 2(15.0142) + 4.992) = \frac{1}{8}(95.9636) = 11.9955$

Hence, the approximate solution to the system of the equation above includes $x = 15.0142$, $y = 4.992$ and $z = 11.9955$

$\simeq x = 15$, $y = 5$ and $z = 12$ respectively

5a) To find the exact solution for $\frac{dy}{dx} = xy$, we use the variable separable method;

$\Rightarrow \frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \int \frac{dy}{y} = \int x\,dx = \ln y = \frac{x^2}{2} + C$

$x_0 = 0$ and $y_0 = 1$

$\Rightarrow \ln y = \frac{x^2}{2} + C \Rightarrow \ln y_0 = \frac{x_0^2}{2} + C \Rightarrow C = 0$

$\Rightarrow \ln y = \frac{x^2}{2}$ or $y = Ae^{x^2/2} \Rightarrow y(x) = 0$

Hence, the exact solution of $\frac{dy}{dx} = xy \Rightarrow \ln y = \frac{x^2}{2}$

Now, Using the picard's iteration methods.

$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad x = 0$

First iteration $n=1$, $x_0 = 0$

$y_1(x) = 0 + \int_0^x (xy_0)\,dx = 0 + \int_0^x x\,dx = 0 + \frac{x^2}{2}\Big|_0^x = 0 + \frac{x^2}{2}$,

At the initial condition, $x_0 = 0$, $y_0(x) = 0$

$y(x) = 0.$

Percentage Error is a way of measuring how far an approximate value is from the exact value.

$$P.E = \frac{|\text{Exact value} - \text{Approximate value}|}{|\text{Exact value}|} \times 100\%.$$

Hence, the p.E mor will be;

$$p \cdot E = \frac{|0-0|}{0} \times 100\% = \underline{0\%}$$

(5b) $y' = 3x + \frac{y}{2}$ subject to initial condition, $y(0) = 1$.

x	Euler's method	Improved Euler's	Exact solution $y(x) = 13e^{x/2} - 6x - 12$
0.0	1.000	1.000	1.000
0.1	1.050	1.066	1.0666
0.2	1.133	1.149	1.1491
0.3	1.249	1.267	1.2669
0.4	1.402	1.420	1.4203
0.5	1.592	1.611	1.6113

(6a) $\frac{dy}{dx} = x + y$ with $y(0) = 1$

$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \Rightarrow$ first iteration $x_0 = 0$, $y_0 = 1$

$y_1(x) = 1 + \int_0^x (x+1)\,dx = 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + \frac{x^2}{2} + x$

Second iteration $n = 2$,

$y_2(x) = 1 + \int_0^x \left(x + 1 + \frac{x^2}{2} + x\right)dx = 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2}\right)dx = 1 + \left[x + x^2 + \frac{x^3}{6}\right]_0^x$

$= 1 + x + x^2 + \frac{x^3}{6}$

Third iteration, $n = 3$

$y_3(x) = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right)dx = 1 + \left[x^2 + x + \frac{x^3}{3} + \frac{x^4}{24}\right]_0^x$

$= 1 + x^2 + x + \frac{x^3}{3} + \frac{x^4}{24}$

Forth iteration, $n = 4$

$y_4(x) = 1 + \int_0^x \left(x + x^2 + 1 + x + \frac{x^3}{3} + \frac{x^4}{24}\right)dx = 1 + \left[\frac{x^3}{3} + x^2 + x + \frac{x^4}{12} + \frac{x^5}{120}\right]_0^x$

$= 1 + \frac{x^3}{3} + x^2 + x + \frac{x^4}{12} + \frac{x^5}{120}$.

Hence, $y(0.2) \Rightarrow n=2 \Rightarrow y_2(x) = 1 + x + x^2 + \frac{x^3}{6}$

$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{6} = 1.2 + 0.04 + 0.001333 = \underline{\underline{1.2413}}$

$y(0.4) \Rightarrow n=3 \Rightarrow y_3(x) = 1 + x^2 + x + \frac{x^3}{3} + \frac{x^4}{24}$

$y(0.4) = 1 + (0.4)^2 + (0.4) + \frac{(0.4)^3}{3} + \frac{(0.4)^4}{24} = 1.4 + 0.16 + 0.00373 = 1.563$

$y(0.6) \Rightarrow n=4 \Rightarrow y_4(x) = 1 + \frac{x^3}{3} + x^2 + x + \frac{x^4}{12} + \frac{x^5}{120}$

$y(0.6) = 1 + 0.6^2 + 0.6 + \frac{(0.6)^3}{3} + \frac{(0.6)^4}{12} + \frac{(0.6)^5}{120} = 2.0434$

(6b) Using Adam-Bashforth predictor - corrector method:

$y_4^p = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$ & $y_4^c = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1']$

$y_0' = x+y \Rightarrow y_0' = 0+1 = 1$, $y_1' = 0.2 + 1.2413 = 1.4413$,

$y_2' = 0.4 + 1.5637 = 1.9637$, $y_3' = 0.6 + 2.0434 = 2.6434$

$\Rightarrow y_4^p = 2.0434 + \frac{0.2}{24}[55(2.6434) - 5(1.9637) + 3(1.4413) - 9(1)]$

$= 2.0434 + 1.09077 = 3.13417$

$\Rightarrow y_4' = 0.8 + 3.13417 = 3.93417$

$y_4^c = 2.0434 + \frac{0.2}{24}[9(3.93417) + 19(2.6434) - 5(1.9637) + 1.4413]$

$= 2.0434 + 0.6438 = 2.6872 \Rightarrow y_4' = 0.8 + 2.6872 = 3.4872$

n	x	y	y' = x+y
0	0	1	1
[illegible]	0.2	1.2413	1.4413
[illegible]	0.4	1.5637	1.9637
[illegible]	0.6	2.0434	2.6434
4	0.8	Predictor 3.13417	3.94317
		Corrector 2.6872	3.4872